Order of Grover's search algorithm

with both total and local depolarizing channel error

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Introduction

Noise is an inherent part of quantum computation. Although there exists a fault-tolerant approach to quantum computing [1], this requires many computational resources. Hence, it is important to analyze how noise affects well-known algorithms. In this paper the effect of noise in Grover's search algorithm [2] is studied. The noise is modeled as both total depolarizing channel (TDCh), and local depolarizing channel (LDCh) in every qubit [3].

An analysis of the order has been made analytically for the TDCh, and an approximation has been done for the LDCh.

Grover's Search Algorithm

Grover's quantum search algorithm [2], solves the problem of finding a marked element (the target state $|t\rangle$) in an unsorted database of N elements ($N=2^n$), in $k_{Gr}=\left\lfloor \frac{\pi}{4}\sqrt{N} \right\rfloor$ steps, where n is the number of qubits. The search is initialized with the superposition state $|s\rangle = H^{\otimes n}|0\rangle$. The probability of finding the target after applying the operator $G=\left(I-2|s\rangle\langle s|\right).\left(I-2|t\rangle\langle t|\right)$, k times is [2]:

$$p_k = \sin^2\left[(2k+1)\arcsin\left(\frac{1}{\sqrt{N}}\right)\right]. \tag{1}$$

TDCh error model in Grover's algorithm

The Total Depolarizing Channel (TDCh) acts in an n-qubit input state ρ, defined by the operator: $\varepsilon(\rho,\gamma)=\gamma\frac{I}{N}+(1-\gamma)\rho$, where γ is the probability of error. Applying this error in every step of the algorithm yields the state $\hat{\rho}_k^T = (1\text{-}\gamma)^k\rho_k+(1-(1\text{-}\gamma)^k)\frac{I}{N}$, where k is the number of steps and $\rho_k = (G)^k|s\rangle\langle s|(G^{+})^k$ is Grover's algorithm original state (without error). Thus, the probability of finding the target state is

$$\hat{p}_k^T(\gamma) = (1\text{-}\gamma)^k p_k+(1-(1\text{-}\gamma)^k)\frac{1}{N}, \tag{2}$$

where p_k is the probability given by eq. (1). Deriving the eq. (2), the step in which the probability is maximum is found:

$$k_{\max} = \max\left(\left\lfloor \frac{\pi-\arcsin(\delta)-\arcsin\left(\left[\left(1-\frac{2}{N}\right)\delta\right]\right)}{4\theta}\right\rfloor, 1\right), \tag{3}$$

where $\theta = \arcsin\left(\frac{1}{\sqrt{N}}\right)$ and $\delta = \frac{1}{\sqrt{1+\left(\frac{4\theta}{\ln(1-\gamma)}\right)^2}}$. A similar approach was taken in [4].

For $n >> 1$ and $0 \le \gamma << \frac{2\pi}{\sqrt{N}}$, the Taylor approximation of eq. (3) is $k_{\max} \approx \left\lfloor \frac{\pi}{4}\sqrt{N} - \frac{N\gamma}{8} \right\rfloor$.

Stopping the algorithm after k_{max} steps, the order with TDCh error becomes:

$$O \equiv \frac{k_{\max}}{\hat{p}_{k_{\max}}} \approx \frac{\pi\sqrt{N}}{4}\left(1+\frac{\pi\sqrt{N}}{4}\left(1-\frac{2}{\pi^2}\right)\gamma\right). \tag{4}$$

We conclude, as in [4], that for γ constant the algorithm becomes of order *N*.

The LDCh error model in Grover's algorithm

The Local Depolarizing Channel (LDCh), acts in an n-qubit state ρ applying the depolarizing channel in every qubit independently ($\varepsilon(\rho,\alpha)=\varepsilon_1(\rho,\alpha)\otimes\cdots\otimes\varepsilon_n(\rho,\alpha)$, where $\varepsilon_i(\rho,\alpha)$ corresponds to the depolarizing channel acting in the qubit i).

We propose both lower and upper bounds to $\hat{p}_k^L(\alpha)$ (the probability of finding the target in k steps in Grover's algorithm with LDCh error) in terms of the TDCh, deduced from LDCh's Kraus operators. These are: $\gamma_l = 1-\frac{(4-3\alpha)^n-\alpha^n}{4^n}$ and $\gamma_u = \frac{n\alpha}{2+n\alpha}$.

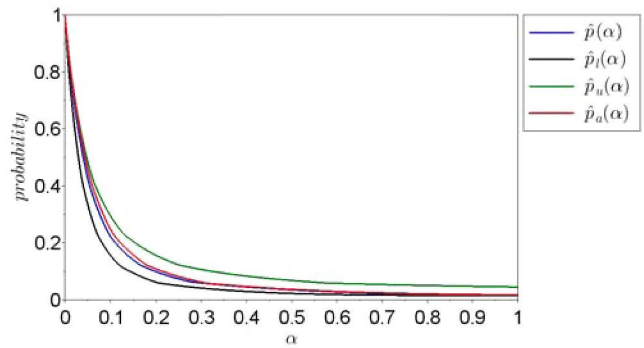

Figure 1: Probabilities of finding the target state for n = 6 at k_{max}. Simulation with LDCh error (blue), lower and upper bounds (black and green) and the proposed approximation (red).

The relationship $\hat{p}_k^T(\gamma_l(\alpha)) \le \hat{p}_k^L(\alpha) \le \hat{p}_k^T(\gamma_u(\alpha))$ holds at least numerically up to 8 qubits. We also propose an approximation of $\hat{p}_k^L(\alpha)$, trying to minimize $\left|\hat{p}_k^T(\gamma_a(\alpha)) - \hat{p}_k^L(\alpha)\right|$, given by $\gamma_a = 1-\left(1-\frac{\alpha}{2}\right)^n$, as seen in Fig 1.

Finally, the obtained orders for $\alpha << 1$ and $n >> 1$ are: $O \approx \frac{\pi\sqrt{N}}{4}\left(1+\frac{\pi\sqrt{N}}{4}\left(1-\frac{2}{\pi^2}\right)\frac{3\alpha\log_2 N}{4}\right)$ (with γ_l), and $O \approx \frac{\pi\sqrt{N}}{4}\left(1+\frac{\pi\sqrt{N}}{4}\left(1-\frac{2}{\pi^2}\right)\frac{\alpha\log_2 N}{2}\right)$ (with γ_u and γ_a).

References

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